STA 111 Practice Problems

1 Thinking about Estimation

These questions are designed to get you thinking about estimation. For each question report your numeric estimate and justify your reasoning in one or two sentences. Note there is not necessarily a single correct answer.

1.1. Suppose the height in inches of Duke male undergraduates is normally distributed with unknown mean μ and known sd 4. You take a random sample of nine students and observe the values below. What is a reasonable estimate of μ?

$$70.37, 65.24, 63.84, 62.32, 68.77, 64.43, 64.51, 68.95, 66.91$$

1.2. Suppose the number of clicks on a webpage per hour follows a Poisson distribution with unknown parameter λ. You take a random sample and observe the values below. What is a reasonable estimate of λ?

$$26, 17, 11, 16, 17, 10, 12, 14, 11, 14$$

1.3. Suppose you take a random sample from the Uniform$[0, \theta]$ distribution and observe the values below.[1] What is a reasonable estimate of θ?

$$10.65, 11.50, 0.54, 8.14$$

2 Two Estimators

Let $X_1, \ldots, X_n$ be a random sample from the density function below, where $\mathbb{E}[X] = \frac{2}{3}\gamma$ and $\mathbb{E}[X^2] = \frac{1}{2}\gamma^2$. Consider estimating γ with two different estimators, $\hat{\gamma}_1 = \frac{3}{2}\bar{X}$ and $\hat{\gamma}_2 = \max(X_1, \ldots, X_n)$.

$$f(x|\gamma) = \begin{cases} \dfrac{2x}{\gamma^2} & 0 \le x \le \gamma \\ 0 & x > \gamma \end{cases}$$

2.1. What is $\mathbb{E}[\hat{\gamma}_1]$? Is $\hat{\gamma}_1$ biased? If it is, report the bias.

2.2. What is $\text{Var}(\hat{\gamma}_1)$?

2.3. What is $F(x) = \mathbb{P}(X \le x)$?

2.4. What is the density function of $\hat{\gamma}_2$?

2.5. What is $\mathbb{E}[\hat{\gamma}_2]$? Is $\hat{\gamma}_2$ biased? If it is, report the bias.

2.6. Write out a single integral expression for $\text{Var}(\hat{\gamma}_2)$. No need to evaluate.

3 Maximum Likelihood

Let $X_1, \ldots X_n$ be a random sample from the density function below, where $\mathbb{E}[X] = 2/\theta$, and $\text{Var}(X) = 2/\theta^2$.

$$f(x) = \theta^2 x e^{-x\theta} \text{ for } x \ge 0$$

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3.1. What is the MLE of θ?

3.2. What is the MLE of the mean?

[1] The Uniform$[0, \theta]$ distribution has density function $f(x) = \frac{1}{\theta}$ for $0 \le x \le \theta$

4 Poisson

Let $X_1, \ldots X_n$ be a random sample from a Poisson distribution with mean λ, where $\lambda > 0$.

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \text{ for } x = 0, 1, 2, 3, \ldots$$

4.1. What is the maximum likelihood estimator of λ?

4.2. What is the maximum likelihood estimator of $\mathbb{P}(X = 4)$?

4.3. What is the MLE of the standard deviation of the distribution?

5 Poisson or Geometric

5.1. Let X be a discrete random variable with $\theta = 0$ or $\theta = 1$. If $\theta = 0$, X is Poisson with $\lambda = 1$ and if $\theta = 1$, X is Geometric with $p = 0.50$. Suppose we observe $X = 3$. What is the MLE of θ?

6 Percentile MLE

Let $X_1, \ldots, X_n$ be a random sample from the density function below.

$$f(x; \alpha) = \begin{cases} 6^\alpha \alpha x^{-(\alpha+1)} & x \geq 6, \quad \alpha > 0 \\ 0 & x < 6 \end{cases} \qquad F(x; \alpha) = \begin{cases} 1 - \left(\dfrac{6}{x}\right)^\alpha & x \geq 6, \quad \alpha > 0 \\ 0 & x < 6 \end{cases}$$

6.1. What is the MLE of α?

6.2. What is the MLE of the 90^{th} percentile of this distribution (an MLE of κ where $\mathbb{P}(X \leq \kappa) = 0.90$)?

7 Maximum Likelihood

Let $X_1, \ldots X_n$ be a random sample from the density function below, where $\alpha > 0$.

$$f(x) = \alpha x^{\alpha-1} \qquad 0 \leq x \leq 1$$

7.1. Find the maximum likelihood estimator of α.

8 Maximum Likelihood

Suppose $X_1, \ldots, X_n$ are a random sample from a distribution with the probability density function below.

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} \text{ where } x > 0$$

8.1. What is the MLE of σ?

9 Estimation of σ^2

9.1. Is S^2 an unbiased estimator of the population variance σ^2?

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

It may be helpful here to note that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (or $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2$).

9.2. What is an unbiased estimator of σ^2? *Hint: Very little work is required here.*

10 Minimum of Exponential

Let $X_1, \ldots, X_n$ be a random sample of size n from an exponential random variable with rate λ, where $\lambda > 0$. Denote by $X_{(1)}$ the sample minimum, $X_{(1)} = \min(X_1, \ldots, X_n)$.

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

10.1. What is the density function of $X_{(1)}$? *Hint: Start with the distribution function.*

10.2. What is $\mathbb{E}[X_{(1)}]$? *Hint: No calculus is required here.*

10.3. Set up a single integral expression for $\mathrm{Var}(X_{(1)})$. No need to evaluate.

11 MLE

Let $X_1, \ldots, X_n$ be a random sample from the distribution below. Note $\mathbb{E}[X] = (\sigma\sqrt{2})/\sqrt{\pi}$ and $\mathrm{Var}(X) = \sigma^2(1-2/\pi)$.

$$f(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-x^2/2\sigma^2} \quad \text{where } x > 0 \text{ and } \sigma > 0$$

11.1. What is the MLE of σ^2?

11.2. Suppose you estimate σ^2 using the estimator $\tilde{\sigma}^2$ (not the MLE from 11.1). Is $\tilde{\sigma}^2$ unbiased?

$$\tilde{\sigma}^2 = \sum_{i=1}^{n} \frac{X_i^2}{2n}$$

12 Estimation of λ

Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with parameter λ and recall $\mathbb{E}[X] = \mathrm{Var}(X) = \lambda$. Consider the estimators of λ below.

- $\hat{\lambda}_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- $\hat{\lambda}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2$
- $\hat{\lambda}_3 = X_1$

12.1. What is the bias of $\hat{\lambda}_1$?

12.2. What is the variance of $\hat{\lambda}_1$?

12.3. What is the MSE (mean-squared error) of $\hat{\lambda}_3$?

12.4. What is the bias of $\hat{\lambda}_2$?

13 Sample Minimum

Let $X_1, \ldots, X_n$ be a random sample from a distribution with the probability distribution function given below. A reasonable estimator of θ is $\hat{\theta} = X_{(1)} = \min(X_1, \ldots, X_n)$.

$$f(x;\theta) = \begin{cases} \dfrac{\theta}{x^2} & 0 < \theta \leq x \\ 0 & x \leq \theta \end{cases} \qquad F(x;\theta) = \begin{cases} 1 - \dfrac{\theta}{x} & 0 < \theta \leq x \\ 0 & x \leq \theta \end{cases}$$

13.1. What is the cumulative distribution function of $\hat{\theta}$?

13.2. What is the density function of $\hat{\theta}$?

13.3. Is $\hat{\theta}$ unbiased?

14 Another MLE

Let $X_1, \ldots, X_n$ be a random sample with density $f(x) = \beta c^{\beta} x^{-(\beta+1)}$ for $x > c$, with $c > 0$ and $\beta > 0$.

14.1. Find the MLE of β.

15 Time on Exam MLE

Let X denote the time that a students spends working on a STA 111 exam. Suppose the density function of X is given below.

$$f(x;\theta) = (\theta+1)x^{\theta} \text{ for } 0 \leq x \leq \theta \text{ and } -1 < \theta$$

15.1. Let $X_1, \ldots, X_n$ be a random sample of n students. Find the MLE of θ.

16 Yet Another MLE

Let $X_1, \ldots, X_n$ be a random sample from the density function $f(x) = \theta x^{-2}$ for $0 < \theta \leq x < \infty$.

16.1. Find the MLE of θ.

17 Uniform MLE

Let $X_1, \ldots, X_n$ be a sample from the uniform distribution on the interval $[\theta_1, \theta_2]$, where $-\infty < \theta_1 < \theta_2 < \infty$.

17.1. Find the MLE of θ_1

17.2. Find the MLE of θ_2.

18 Lightbulb Lifetimes

Suppose the lifetime of a certain brand of lightbulb has an exponential distribution with guaranteed minimum lifetime θ, as shown below (this is called the shifted exponential distribution). You take a random sample $X_1, \ldots X_n$ of lightbulbs and record their lifetimes.

$$f(x;\lambda,\theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)} & x \geq \theta \\ 0 & x < \theta \end{cases} \qquad F(x;\lambda,\theta) = \begin{cases} 1 - e^{-\lambda(x-\theta)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

18.1. What is the maximum likelihood estimator of θ?

18.2. What is the maximum likelihood estimator of λ?

18.3. Find $\mathbb{E}[\hat{\theta}]$. Is $\hat{\theta}$ unbiased?

18.4. If $n = 7$ lightbulb lifetimes are recorded (shown below), what are the numeric values of $\hat{\theta}$ and $\hat{\lambda}$?

28.54, 22.64, 21.75, 15.02, 11.60, 18.24, 14.13

19 Discrete MLE

You draw a single observation from a discrete random variable X with parameter θ, where $\theta = 1$, 2, or 3. The probability mass functions for $\theta = 1$, $\theta = 2$, and $\theta = 3$ are shown in the table below.

19.1. If $X = 0$, what is the MLE of θ?

19.2. If $X = 1$, what is the MLE of θ?

19.3. If $X = 2$, what is the MLE of θ?

19.4. If $X = 3$, what is the MLE of θ?

19.5. If $X = 4$, what is the MLE of θ?

19.6. Is the MLE unique?

$\theta = 1$		$\theta = 2$		$\theta = 3$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0	0.30	0	0	0	0.10
1	0.30	1	0.25	1	0.15
2	0	2	0.25	2	0.25
3	0.20	3	0.25	3	0.50
4	0.20	4	0.25	4	0

20 Normal MLE

Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, 1)$ distribution.

20.1. What is the MLE for θ?

21 Geometric MLE

Let $X_1, \ldots, X_n$ be a random sample from a Geometric distribution with probability of success p.

$$P(X = x) = (1 - p)^{x-1} p$$

21.1. Find the MLE of p.

22 Uniform MLE

Let $X_1, \ldots X_n$ be iid with pdf given below.

$$f(x) = \frac{1}{\theta} \quad 0 \leq x \leq \theta, \quad \theta > 0$$

22.1. What is the MLE of θ?